AD-A059 853 ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND ABERD--ETC F/G 19/5
SOME BOUNDS FOR OPTIMAL MANEUVERS AND PREDICTORS.(U)
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TECHNICAL REPORT ARBRL-TR-02101

SOME BOUNDS FOR OPTIMAL MANEUVERS AND PREDICTORS

Harry L. Reed, Jr.

September 1978

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: TECHNICAL REPORT ARBRL-TR-02101

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle): SOME BOUNDS FOR OPTIMAL MANEUVERS AND PREDICTORS.

5. TYPE OF REPORT & PERIOD COVERED

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s): Harry L. Reed, Jr.

8. CONTRACT OR GRANT NUMBER(s)

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
US Army Ballistic Research Laboratory
(ATTN: DRDAR-BLB)
Aberdeen Proving Ground, MD 21005

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: RDTE Project 1L162618AH80

11. CONTROLLING OFFICE NAME AND ADDRESS:
US Army Armament Research & Development Command
US Army Ballistic Research Laboratory
(ATTN: DRDAR-BL)
Aberdeen Proving Ground, MD 21005

12. REPORT DATE: SEPTEMBER 1978

13. NUMBER OF PAGES: 33

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

(12) 31p.

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

(18) SBIE (19) AD-E430 111

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Fire Control
Predictors
Optimum Maneuvers
Hit Probability

20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (kst/emj)

A problem of some interest in the understanding of fire control systems is the following pure prediction problem. Given the class of target trajectories $\underline{X}$ for which

$$\lim_{B \to \infty} \frac{1}{2B} \int_{-B}^{B} [\ddot{x}(t)]^2 dt = a^2 \quad \text{where } x \in \underline{X}$$

and the class of predictors P which satisfy the causal principle, find

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

393 471 LB

$$\varepsilon_1 = \sup_{x \in \overline{X}} \inf_{p \in P} \varepsilon$$

and

$$\varepsilon_2 = \inf_{p \in P} \sup_{x \in \overline{X}} \varepsilon$$

where

$$\varepsilon_2 = \frac{1}{2B} \int_{-B}^{B} [x(t+T) - y(t+T)]^2 dt$$

and $y(t+T)$ is the predicted value of $x(t+T)$ based on values of $x(t-\tau)$ with $\tau \geqslant 0$.

It is shown that

$$\varepsilon_1 = \varepsilon_2$$

for general N. For the particular value of $N = 2$ which corresponds to a limited acceleration for the target, we have

$$\varepsilon_1 = \varepsilon_2 = \frac{2}{(\lambda T)^2} [\tfrac{1}{2} a T^2] \cong 0.569 [\tfrac{1}{2} a T^2]$$

where λ is determined from the solution of an eigenvalue problem for a fourth-order differential equation.

The prediction algorithm p_α for which

$$\varepsilon_1 = \varepsilon(x_\alpha, p_\alpha)$$

is a linear operator and the optimal subclass of maneuvers x_α is based on a second-order correlation function

$$E[\ddot{x}(t)\ddot{x}(t+\tau)] = \int_0^\infty \alpha(s)\alpha(s+\tau)ds.$$

For the particular case of $N = 2$ we have

$$\alpha(s) = \frac{a}{\sqrt{T}} \left[\frac{\cosh \lambda(s - T/2)}{\cosh(\lambda T/2)} - \frac{\sin \lambda(s - T/2)}{\sin \lambda T/2} \right]$$

for

$$0 \leqslant s \leqslant T$$

and

$$\alpha(s) = 0 \text{ otherwise.}$$

No restrictions were placed on $\overline{X}$ and P other than those stated above (i.e., $\overline{X}$ was not restricted to stationary processes and P was not restricted to linear operators).

It is further shown that the strategies given above are good approximations for the more general analysis in which hit probability is the performance measure. This is the case at least for "first cut" analyses.

Bounds such as this help avoid the expenditure of resources to achieve the impossible or to achieve marginally small improvements in fire control design.

TABLE OF CONTENTS

1. INTRODUCTION

The goal of this paper is to give some insights into how well a fire control system can be expected to perform with noiseless information and how well a target can avoid being hit with limits on its ability to maneuver. Such a goal is very ambitious, so we shall make three simplifying assumptions:

- The tracking data are noiseless. This gives an advantage to the gun (see Reference 1) but is not too significant since optical and millimeter radar systems promise very accurate tracking and also since in many cases the errors from evasive maneuvers far exceed errors resulting from errors in state estimation.

- The target is limited only in the r.m.s. value of the N^{th} derivative of its path. That is, the class of maneuvers $\underline{\overline{X}}_N$ is limited to those x(t) for which

$$C_N{}^2 = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} [x^{(N)}(t)]^2 dt. \qquad 1.1$$

- The performance of the fire control is characterized by the r.m.s. prediction error

$$\varepsilon^2(x,p) = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} [\hat{x}(t+T,p) - x(t+T)]^2 dt \qquad 1.2$$

where T is the time of flight of the bullet, $p \in P$, P is the class of prediction algorithm such that $\hat{x}$ is the predicted value of x(t+T) given all data on x(t-s) for $s \geqslant 0$.

The last two assumptions are the closest concession we make to stationarity. The reason for averaging over time in Equation 1.2 is to provide a measure that does not encourage the target to make a one-time maneuver at the time of firing a single round but rather forces the target to avoid rounds fired at unknown times or to avoid bursts of rounds fired over time.

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[1]Harry L. Reed, Jr., "Some Bounds on the Generalized Fire Control Problem," Ballistic Research Laboratories Report No. 1946, November 1976 (AD A033043).

Finally, the use of an r.m.s. error gives an incomplete measure of effectiveness for maneuvers with statistics that do not allow an adequate measure of probability of hit from the r.m.s. error (see, for example, Reference 2). However, for optimal maneuvers, the r.m.s. error is a fairly good measure (see Section 5).

We shall omit the subscript N unless its particular value is important. The following is our overall strategy:

Let
$$\varepsilon_o = \sup_{x \in \underline{\overline{X}}} \ \inf_{p \in P} \varepsilon(x,p) \tag{1.3}$$

$$\varepsilon^o = \inf_{p \in P} \ \sup_{x \in \underline{\overline{X}}} \varepsilon(x,p) \tag{1.4}$$

$$\tilde{\varepsilon}_o = \varepsilon(x_o,p_o) = \sup_{x \in \underline{\overline{X}}_G} \ \inf_{p \in P} \varepsilon(x,p) \tag{1.5}$$

$$\tilde{\varepsilon}^o = \sup_{x \in \underline{\overline{X}}} \ \varepsilon(x,p_o) \tag{1.6}$$

where $\underline{\overline{X}}_G$ is the class of stationary Gaussian maneuvers that satisfy Equation 1.1 and also satisfy

$$\int_{-\infty}^{\infty} \frac{|\log|\Phi(w)||}{1 + w^2} \, dw < \infty \tag{1.7}$$

where $\Phi(w)$ is the power spectral density of the Nth derivative.

Since $\underline{\overline{X}}_G \subset \underline{\overline{X}}$ and $p_o \in P$, we have

$$\tilde{\varepsilon}_o \leqslant \varepsilon_o \leqslant \varepsilon^o \leqslant \tilde{\varepsilon}^o \tag{1.8}$$

The argument that gives the middle inequality is a consequence of the properties of sup and of inf, is common knowledge in game theory, and is given in Appendix 1 for the benefit of the uninitiated.

We shall show that

$$\tilde{\varepsilon}_o = \tilde{\varepsilon}^o \tag{1.9}$$

and thus that

$$\varepsilon_o = \varepsilon^o . \tag{1.10}$$

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[2]Harry L. Reed, Jr., " Limitations of the R.M.S. Criterion for Fire Control," Ballistic Research Laboratories Report No. 1805, July 1975 (AD A014986).

We shall also evaluate ε_o for N = 0, 1, and 2 and show how to evaluate it for higher values of N.

Equation 1.10 implies that in a game between two "smart" players, x_o and p_o are optimal strategies.

2. LOWER BOUND

For Gaussian maneuvers the optimal predictors are linear operators of the form (see Reference 3)

$$\hat{x}(t+T,p_h) = \sum_{m=o}^{N-1} x^{(m)}(t) \frac{T^m}{m!} + \int_o^\infty h(s)x^{(N)}(t-s)ds \qquad 2.1$$

Integration by parts gives

$$\begin{aligned} x(t+T) - \hat{x}(t+T, p_h) &= \int_o^\infty u_N(T-s)x^{(N)}(t+s)ds \\ &\quad - \int_o^\infty h(s)x^{(N)\cdot}(t-s)ds \end{aligned} \qquad 2.2$$

where

$$u_o(t) = \delta(t) \qquad 2.3$$

and for $N > 0$

$$u_N(t) = \frac{t^{N-1}}{(N-1)!} \quad \text{for } t \geqslant 0 \qquad 2.4$$

$$= 0 \qquad \text{for } t < 0. \qquad 2.5$$

Again we shall use

$f(t)$ for $u_N(T-t)$

and

$a(t)$ for $x^{(N)}(t)$

unless the particular value of N is important to the argument at hand.

---

[3] Norbert Wiener, *Extrapolation, Interpolation, and Smoothing of Stationary Time Series*, The Technology Press of M.I.T. and John Wiley & Sons, Inc., New York.

Let

$$\phi(s) = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} a(t)a(t+s)dt. \qquad 2.6$$

Equations 1.1 and 1.7 allow us to write

$$\phi(s) = \int_{-\infty}^{\infty} \alpha(t)\alpha(t+s)dt \qquad 2.7$$

where

$$\alpha(t) = 0 \qquad t < 0 \qquad 2.8$$

and of course

$$C^2 = \int_{0}^{\infty} [\alpha(t)]^2 dt. \qquad 2.9$$

Using

$$\phi(r-s) = \int_{-\infty}^{\infty} \alpha(t+r)\alpha(t+s)dt \qquad 2.10$$

$$= \int_{-\infty}^{\infty} \alpha(t-r)\alpha(t-s)dt \qquad 2.11$$

and

$$\phi(r+s) = \int_{-\infty}^{\infty} \alpha(t-r)\alpha(t+s)dt, \qquad 2.12$$

we can combine Equation 1.2 and 2.2 to write

$$\varepsilon^2 = \int_{-\infty}^{\infty} dt \left\{ \int_{0}^{\infty} [f(s)\alpha(t+s) - h(s)\alpha(t-s)]ds \right\}^2 \qquad 2.13$$

$$= \int_{0}^{\infty} dt \left\{ \int_{0}^{\infty} [f(s)\alpha(t+s) - h(s)\alpha(t-s)]ds \right\}^2$$

$$+ \int_{-\infty}^{0} dt \left\{ \int_{0}^{\infty} f(s)\alpha(t+s)ds \right\}^2 . \qquad 2.14$$

To minimize with respect to p, we pick h to satisfy

$$\int_0^\infty f(s)\alpha(t+s)ds = \int_0^\infty h(s)\alpha(t-s)ds \qquad 2.15$$

which puts the first term of the function in Equation 2.14 equal to zero.

To maximize with respect to x, we then pick α to maximize

$$\varepsilon^2 = \int_{-\infty}^0 dt \left\{\int_0^\infty f(s)\alpha(t+s)ds\right\}^2$$

$$= \int_0^\infty dt \left\{\int_0^\infty f(s)\alpha(s-t)ds\right\}^2 \qquad 2.16$$

$$= \int_0^\infty dt \left\{\int_0^\infty f(s+t)\alpha(s)ds\right\}^2 .$$

To do this, we set

$$\delta\varepsilon^2 = 2\int_0^\infty dt \int_0^\infty f(s+t)\alpha(s)ds \int_0^\infty f(r+t)\delta\alpha(r)dr = 0 \qquad 2.17$$

subject to

$$\int_0^\infty \alpha(r)\ \delta\alpha(r) = 0. \qquad 2.18$$

Therefore

$$\alpha(r) = k\int_0^\infty dt \int_0^\infty f(r+t)\ f(s+t)\alpha(s)ds. \qquad 2.19$$

Multiplying Equation 2.19 by $\alpha(r)$ and integrating, we have

$$\varepsilon^2 = \frac{C^2}{k} \qquad 2.20$$

which shows that $k \geqslant 0$. Further

$$\varepsilon_o{}^2 = \frac{C^2}{k_o}$$

where k_o is the least eigenvalue of Equations 2.9 and 2.19.

In Section 4 we show how this eigenvalue problem is related to an eigenvalue problem for a system of differential equations and we evaluate k_o for N = 0, 1, and 2.

3. UPPER BOUND

Even though the class $\underline{\overline{X}}$ is only constrained by

$$C^2 = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} [a(t)]^2 dt$$

we can define

$$\phi(s) = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} a(t)a(t+s)dt \qquad 3.1$$

and know that

$$\Phi(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(s)e^{-iws}ds \geqslant 0. \qquad 3.2$$

Now we shall use the predicter which was defined in the previous section by Equation 2.15 for the particular $\alpha(r)$ given in Equation 2.19. Then Equations 1.7, 2.2, and 3.1 give (for any x)

$$\varepsilon^2(x,p_o) = \int_0^{\infty} dr \int_0^{\infty} ds \left\{ f(s)f(r)\phi(r-s) - 2\, f(s)h(r)\phi(r+s) + h(r)h(s)\phi(r-s) \right\}. \qquad 3.3$$

Using

$$\phi(r) = \int_{-\infty}^{\infty} \Phi(w)e^{iwr}dw, \qquad 3.4$$

we can derive

$$\varepsilon^2(x,p_o) = \int_{-\infty}^{\infty} |\ \overline{F}(w) - H(w)\ |^2\ \Phi(w)dw \qquad 3.5$$

where

$$H(w) = \int_0^{\infty} h(t)e^{-iwt}dt \tag{3.6}$$

and

$$F(w) = \int_0^{\infty} f(t)e^{-iwt}dt \tag{3.7}$$

We shall show that

$$|\overline{F}(w) - H(w)|^2 = \frac{1}{k_o} \tag{3.8}$$

and thus that

$$\varepsilon^2(x,p_o) = \frac{1}{k_o}\int_{-\infty}^{\infty} \Phi(w)dw = \frac{C^2}{k_o} \tag{3.9}$$

for all $x \in \underline{\overline{X}}$.

To do that, we first take the Fourier transform of Equation 2.15 to get

$$\int_0^{\infty} e^{-iwt}dt \int_0^{\infty} f(s)\alpha(t+s)ds = H(w)A(w) \tag{3.10}$$

where

$$A(w) = \int_0^{\infty} \alpha(t)e^{-iwt}dt \tag{3.11}$$

Some manipulation of the left hand side of Equation 3.10 gives

$$\begin{aligned}
&\int_0^{\infty} e^{-iwt}dt \int_0^{\infty} f(s)\alpha(t+s)ds \\
&= \int_{-\infty}^{\infty} e^{-iwt}dt \int_0^{\infty} f(s)\alpha(t+s)ds - \int_{-\infty}^{0} e^{-iwt}dt \int_0^{\infty} f(s)\alpha(t+s)ds \\
&= \overline{F}(w)A(w) - \int_0^{\infty} e^{iwt} \int_0^{\infty} f(s)\alpha(s-t)ds \\
&= \overline{F}(w)A(w) - \int_0^{\infty} e^{iwt} \int_0^{\infty} f(t+r)\alpha(r)dr
\end{aligned} \tag{3.12}$$

Therefore

$$|\overline{F}(w) - H(W)|^2 = |B(w)/A(w)|^2 \qquad 3.13$$

where

$$B(w) = \int_{-\infty}^{\infty} e^{iwt}\beta(t)dt \qquad 3.14$$

$$\begin{aligned}\beta(t) &= \int_{0}^{\infty} f(t+r)\alpha(r)rt \qquad t \geqslant 0 \\ &= 0 \qquad\qquad\qquad\qquad\quad t < 0\end{aligned} \qquad 3.15$$

We then have

$$|B(w)|^2 = B(w)\overline{B}(w) = \int_{-\infty}^{\infty} e^{iwp}dp \int_{0}^{\infty} \beta(q)\beta(p+q)dq \qquad 3.16$$

Note that this convolution is an even function of p so that

$$|B(w)|^2 = \int_{-\infty}^{\infty} e^{iwp}dp \int_{0}^{\infty} \beta(q)\beta(|p|+q)dq \qquad 3.17$$

Now using Equation 2.19, we have

$$\begin{aligned}\int_{0}^{\infty} \beta(q)\beta(|p|+q)dq &= \int_{0}^{\infty} dq \int_{0}^{\infty} f(q+r)\alpha(r)dr \\ &\quad \times \int_{0}^{\infty} f(q+|p|+s)\alpha(s)ds \\ &= \frac{1}{k_o}\int_{0}^{\infty} \alpha(|p|+s)\alpha(s)ds \qquad 3.18\end{aligned}$$

and finally

$$|B(w)\bar{B}(w)| = \frac{1}{k_o}\int_{-\infty}^{\infty} e^{iwp}dp \int_0^{\infty} \alpha(|p|+s)\alpha(s)ds$$

$$= \frac{1}{k_o} \; |A(w)\bar{A}(w)| \qquad 3.19$$

and we have from Equation 3.13 and 3.5 that

$$\varepsilon^2(x,p_o) = \frac{C^2}{k_o}$$

as advertised in Equation 3.9.

Another approach to Equation 3.8 is to show that

$$B(w) = \frac{1}{\sqrt{k_o}} \bar{A}(w) \qquad 3.20$$

which can be shown by showing that

$$\beta(t) = \frac{1}{\sqrt{k_o}} \alpha(t). \qquad 3.21$$

To do this, we combine Equation 2.19 and 3.15 to get

$$\beta(t) = \int_0^{\infty} f(t+r)dr \int_0^{\infty} k_o dq \int_0^{\infty} f(r+q)f(s+q)\alpha(s)ds$$

$$= k_o\int_0^{\infty} f(t+r)dr \int_0^{\infty} f(r+q)\beta(q)dq \qquad 3.22$$

Thus $\beta(t)$ satisfies the same integral equation as $\alpha(t)$. In the next section we relate this integral equation to the eigenvalue problem for a differential equation. This eigenvalue problem has only one linearly independent solution (see Appendix 2) and so we can write

$$\beta(t) = \gamma \; \alpha(t) \qquad 3.23$$

Then

$$\int_0^\infty [\beta(t)]^2 dt = \gamma^2 \int_0^\infty [\alpha(t)]^2 dt \qquad 3.24$$

which gives

$$\int_0^\infty \left\{ \int_0^\infty f(t+r)\alpha(r)dr \right\}^2 dt = \gamma^2 \int_0^\infty [\alpha(t)]^2 dt \qquad 3.25$$

using the definition of $\beta(t)$ and which can be rewritten as

$$\begin{aligned} &\int_0^\infty \alpha(r)dr \int_0^\infty f(t+r)dt \int_0^\infty f(t+s)\alpha(s)ds \\ &= \gamma^2 \int_0^\infty [\alpha(t)]^2 dt \end{aligned} \qquad 3.26$$

and finally (from Equation 2.19)

$$\frac{1}{k_o} \int_0^\infty [\alpha(r)]^2 dr = \gamma^2 \int_0^\infty [\alpha(t))]^2 dt \qquad 3.27$$

which gives

$$\gamma = \frac{1}{\sqrt{k_o}} \qquad 3.28$$

4. THE EIGENVALUE PROBLEM

We have

$$\int_0^\infty \alpha^2(r) = C^2 \qquad 4.1$$

and

$$\begin{aligned} \alpha(r) &= k_o \int_0^\infty dt \int_0^\infty u(T-r-t)u(T-s-t)\alpha(s)ds \\ &= k_o \int_0^{T-r} dt \int_0^{T-t} \frac{(T-r-t)^{N-1}}{(N-1)!} \frac{(T-s-t)^{N-1}}{(N-1)!} \alpha(s)ds \end{aligned} \qquad 4.2$$

for $r \leqslant T$.

We also have $k_o > 0$, and k_o is the least eigenvalue of this system of equations.

If $r > T$

$$u(T-r-t) = 0 \quad \text{since} \quad t \geqslant 0$$

and thus

$$\alpha(r) = 0 \quad \text{for} \quad r > T \tag{4.3}$$

If $N = 0$

$$u(r) = \delta(r)$$

and

$$\alpha(r) = k_o \int_o^\infty dt \int_o^\infty \delta(T-r-t)\delta(T-s-t)\alpha(s)ds$$

$$= k_o\,\alpha(r)$$

So for $N = 0$

$$k_o = 1 \tag{4.4}$$

and

$$\varepsilon_o = C \tag{4.5}$$

Let $N \geqslant 1$. We can differentiate Equation 4.2 to get for $0 \leqslant t \leqslant T$

$$\alpha^{(2N)} = (-1)^N k_o \alpha \tag{4.6}$$

$$\alpha^{(M)}(T) = 0 \quad \text{for} \quad M=0 \text{ to } N-1 \tag{4.7}$$

$$\alpha^{(M)}(0) = 0 \quad \text{for} \quad M=N \text{ to } 2N-1 \tag{4.8}$$

The uniqueness of the solution to this system of equations is shown in Appendix 2.

Now let N=1

$$\ddot{\alpha} = -\,k_o\alpha$$

$$\dot{\alpha}(o) = \alpha(T) = 0$$

$$\alpha = \sqrt{2/T}\ C \cos\left(\frac{\Pi}{2T}\ t\right) \qquad 4.9$$

$$k_o = \left(\frac{\Pi}{2T}\right)^2 \qquad 4.10$$

$$\varepsilon_o = \frac{2}{\Pi}\ CT \qquad 4.11$$

Finally let N = 2

$$\ddddot{\alpha} = k_o \alpha$$

$$\dddot{\alpha}\ (o) = \ddot{\alpha}(o) = \dot{\alpha}(T) = \alpha(T) = 0$$

(The classical problem of the vibration of a clamped rod.)

Letting $k_o = \lambda_o{}^4$, we have

$$1 + \cosh\ (\lambda_o T)\ \cos\ (\lambda_o T) = 0 \qquad 4.12$$

$$\lambda_o T \cong 1.875$$

$$\alpha = \frac{C}{\sqrt{T}} \left\{ \frac{\cosh[\lambda_o(t-T/2)]}{\cosh[\lambda_o T/2]} - \frac{\sin[\lambda_o(t-T/2)]}{\sin[\lambda_o T/2]} \right\} \qquad 4.13$$

$$\varepsilon_o = \frac{2}{\left(\lambda_o T\right)^2} \left(\frac{1}{2}\ CT^2\right) \qquad 4.14$$

$$\cong .569 \left(\frac{1}{2}\ CT^2\right) \qquad 4.15$$

5. THE GENERAL PROBLEM

In this section we shall see how far we can go using hit probability rather than the r.m.s. criterion. In doing this, we shall have to give up the neatness of finding an exact answer. On the other hand, we shall find bounds on the problem, and these bounds will be shown to bracket the problem closely enough for many "first analyses."

Let us discuss the problem where x(t) has a single spatial dimension. Associated with a class of maneuvers is a probability distribution function

$$u\{x(t+T) - \hat{x}(t+T,p) \mid x(t-s),\ s \geq 0\}. \tag{5.1}$$

That is, u is the distribution of the error between the future position and the predicted future position given the past. Let y be this error in future position. We can average over time to find a distribution function

$$u(y \mid x,p). \tag{5.2}$$

The probability of hit q is

$$q(x,p) = \int_{-\ell/2}^{\ell/2} du(\xi \mid x,p), \tag{5.3}$$

where ℓ is the size of the target. The pilot wishes to keep q small. His goal might be

$$q^o = \inf_{x \in X} \ \sup_{p \in P} \ q(x,p). \tag{5.4}$$

Likewise, the gunner might try for

$$q_o = \sup_{p \in P} \ \inf_{x \in X} \ q(x,p). \tag{5.5}$$

The set $\underline{X}$, the set P, the set X_G, the maneuver $\underline{X}_o$, the prediction algorithm p_o, and the error ε_o are as defined in Section 1.

We can define

$$\tilde{q}^o = \tilde{q}^o(\tilde{\varepsilon}_o/\ell) = q(x_o,p_o) = \frac{1}{\sqrt{2\pi}\,\tilde{\varepsilon}_o} \int_{-\ell/2}^{\ell/2} e^{-\xi^2/(2\tilde{\varepsilon}_o^{\,2})}\, d\xi \tag{5.6}$$

Since x_o is Gaussian, p_o maximizes the hit probability as well as it minimizes the error. Thus

$$q^o \leq \tilde{q}^o . \qquad 5.7$$

We can also define

$$\tilde{q}_o = \inf_x q(x,\tilde{p}_o) , \qquad 5.8$$

where $\tilde{p}_o$ is a variant of the algorithm p_o and will be described in Section 6.

As usual, we have

$$\tilde{q}_o \leq q_o \leq q^o \leq \tilde{q}^o , \qquad 5.9$$

but this time we have not been able to collapse this chain of inequalities. In fact we are only able to find a lower bound $z(\tilde{\varepsilon}_o/\ell)$ such that

$$z \leq \tilde{q}_o \leq q_o \leq q^o \leq \tilde{q}^o . \qquad 5.10$$

Nevertheless, these bounds may well still be useful for first estimates since they provide a variation of no more than 70 percent. A tabulation of the lower bound z and the upper bound $\tilde{q}^o$ and their ratio is given in Table 1.

6. THE ALGORITHM $\tilde{p}_o$

Define the algorithm $\tilde{p}_o$ to be

$$\hat{x}(t+T,\tilde{p}_o) = y_o + \hat{x}(t+T,p_o). \qquad 6.1$$

Then

$$u(y \mid x,\tilde{p}_o) = u(y - y_o \mid x,p_o) . \qquad 6.2$$

The value y_o is the value that maximizes

$$q(y_o,x,p_o) = \int_{y=-\ell/2}^{y=\ell/2} du(y - y_o) \mid x,p_o) \qquad 6.3$$

From Appendix C we have

Table 1

$z(\varepsilon)$	ε	$\tilde{q}^{o}(\varepsilon)$	$\tilde{q}^{o}/z$
.00	∞	.000	1.382
.05	5.557	.072	1.434
.10	2.669	.149	1.486
.15	1.721	.229	1.524
.20	1.225	.317	1.585
.25	.935	.407	1.628
.30	.775	.481	1.605
.35	.622	.578	1.652
.40	.548	.639	1.597
.45	.461	.722	1.604
.50	.354	.843	1.685
.55	.335	.864	1.571
.60	.316	.886	1.477
.65	.296	.909	1.399
.70	.274	.932	1.332
.75	.250	.954	1.273
.80	.224	.975	1.218
.85	.194	.990	1.165
.90	.158	.998	1.109
.95	.112	1.000	1.053
1.00	.000	1.000	1.000

$$q(x,\tilde{p}_o) = \sup_{y_o} q(y_o,x,p_o) \geq z(\sigma/\ell) \geq z(\tilde{\varepsilon}_o/\ell) \qquad 6.4$$

where σ is the standard deviation around the mean. The last inequality follows since z is a monotonically decreasing function and since σ minimizes the r.m.s. error.

Since the middle inequality in Equation 6.4 holds for all x, we have

$$\tilde{q}_o \geq z(\tilde{\varepsilon}_o/\ell). \qquad 6.5$$

7. CONCLUSIONS

With respect to the r.m.s. criterion and the r.m.s. bound on an Nth derivative, the duel between a gunner and a target is a game with a saddle point which can be precisely defined and hence stable strategies exist for both players.

If hit probability is used as the criterion, we have been unable to define a saddle point precisely. However, we can find bounds that show that the difference between the performance for such a saddle point and the saddle point for the r.m.s. case may well be small enough to use the r.m.s. criterion as a good "first analysis."

APPENDIX A

First consider ε_o. We note that for each $\delta \geqslant 0$ there exists an x_δ such that

$$\inf_p \varepsilon(x_\delta, p) \geqslant \varepsilon_o - \delta$$

which follows from the definition of sup.

Thus

$$\varepsilon(x_\delta, p) \geqslant \varepsilon_o - \delta$$

for all p which follows from the definition of inf.

Likewise, there exists a p_δ such that

$$\varepsilon(x, p_\delta) \leqslant \varepsilon^o + \delta$$

for all x.

Thus

$$\varepsilon_o - \delta \leqslant f(x_\delta\, p_\delta) \leqslant \varepsilon^o + \delta$$

for all δ and thus

$$\varepsilon_o \leqslant \varepsilon^o.$$

APPENDIX B*

Statement of Problem.

Let $k > 0$ be such that the differential equation

$$\begin{aligned} x^{(2n)} &= (-1)^n kx \\ x^{(m)}(T) &= 0 \qquad (m=0,1,\ldots,n-1) \\ x^{(m)}(0) &= 0 \qquad (m=n,n+1,\ldots,2n-1) \end{aligned}$$

has a nontrivial solution. Prove that this solution is unique up to constant multiples.

Proof.

The proof consists in showing that for any two nontrivial solutions x and y the equality sign in Schwarz's inequality

$$\left(\int_0^T xy\ dt\right)^2 \leq \int_0^T x^2 dt \int_0^T y^2 dt$$

holds, which occurs if and only if $y = cx$, c a constant.

To this end, we note first that

$$\int_0^T xy\ dt = \frac{1}{k} \int_0^T x^{(n)} y^{(n)} dt. \tag{B-1}$$

This follows from

---

*The analysis in this appendix was provided by Mr. Walter O. Egerland of the Ballistic Modeling Division, Ballistic Research Laboratory.

$$\int_0^T xy\,dt = \frac{(-1)^n}{k}\int_0^T x^{(2n)}y\,dt$$

$$= \frac{(-1)^n}{k}\left\{ yx^{(2n-1)}\Big|_0^T - \int_0^T x^{(2n-1)}y'dt \right\}$$

$$= \frac{(-1)^{n+1}}{k}\int_0^T x^{(2n-1)}y'dt$$

$$= \frac{(-1)^{n+j}}{k}\int_0^T x^{(2n-j)}y^j dt, \; j=1,2,\ldots n \qquad \text{(B-2)}$$

Next, we write the identify

$$xy = x(0)y(0) + \int_0^t (x'y + xy')dt,$$

and find successively

$$xy = x(0)y(0) + \frac{(-1)^n}{k}\int_0^t (x'y^{(2n)} + y'x^{(2n)}dt$$

$$= x(0)y(0) + \frac{(-1)^n}{k}\Big\{ y^{(2n-1)}x' + x^{(2n-1)}y'$$

$$- \int_0^t (y^{(2n-1)}x'' + x^{(2n-1)}y'')dt \Big\}$$

$$= \vdots$$

$$= x(0)y(0) + \frac{(-1)^n}{k}\left[\sum_{j=1}^{n-1} (-1)^{j+1}\left\{ y^{(2n-j)}x^j + x^{(2n-j)}y^j \right\} + (-1)^{n+1}\, x^{(n)}\, y^{(n)}\right]$$

Integration over the interval [0,T], using (B-1) and (B-2), yields

$$\int_0^T xy\, dt = x(0)y(0)T - (2n - 1)\int_0^T xy\, dt$$

or

$$\int_0^T xy\, dt = \frac{x(0)y(0)}{2n}\, T. \qquad \text{(B-3)}$$

In particular,

$$\int_0^T x^2 dt = \frac{x(0)^2}{2n}\, T$$

and

$$\int_0^T y^2 dt = \frac{y(0)^2}{2n}\, T.$$

Hence, equality in Schwarz's inequality holds, and the proof is complete.

APPENDIX C

Let $u(\xi)$ be any distribution function which we will take for convenience as having zero mean. Then

$$\sigma^2 = \int_{-\infty}^{\infty} \xi^2 \, du(\xi). \tag{C-1}$$

We want to relate σ with

$$q = \sup_y \int_{y-\ell/2}^{y+\ell/2} du \tag{C-2}$$

It is convenient to find a function such that

$$\sigma \geqslant r(q). \tag{C-3}$$

This function is monotonically decreasing with q and thus we can use it to define

$$q \geqslant z(\sigma). \tag{C-4}$$

implicitly.

We can write

$$\sigma^2 = \sum_{m=-\infty}^{\infty} \int_{m\ell/2}^{(m+1)\ell/2} \xi^2 \, du(\xi)$$

$$\sigma^2 \geqslant \sum_{m=0}^{\infty} \left(\frac{m\ell}{2}\right)^2 \int_{m\ell/2}^{(m+1)\ell/2} du + \sum_{m=-\infty}^{-1} \left(\frac{m+1}{2}\right)^2 \int_{m\ell/2}^{(m+1)\ell/2} du$$

$$\sigma^2 \geqslant \sum_{m=0}^{\infty} \left(\frac{m\ell}{2}\right)^2 \mu_m + \sum_{m=-\infty}^{-1} \left[\frac{(m+1)\ell}{2}\right]^2 \mu_m \tag{C-5}$$

where

$$\mu_m = \int_{m\ell/2}^{(m+1)\ell/2} du \geqslant 0,$$

and $$\mu_m + \mu_{m+1} \leqslant q$$

and $$\sum_{m=-\infty}^{\infty} \mu_m = 1 \ .$$

If we write

$$\nu_m = \mu_m + \mu_{(-1-m)} \ ,$$

we have

$$\sigma^2 \geqslant \Sigma = \sum_{m=0}^{\infty} \left(\frac{m\ell}{2}\right)^2 \nu_m \ , \qquad \text{(C-6)}$$

where

$$0 \leqslant \nu_o \leqslant q \ , \qquad \text{(C-7)}$$

$$0 \leqslant \nu_m + \nu_{m+1} \leqslant 2q \ , \qquad \text{(C-8)}$$

$$\sum_{m=0}^{\infty} \nu_m = 1. \qquad \text{(C-9)}$$

It is not hard to show that we have a lower bound for Σ defined in Equation C-6 if we let

$$\nu_m = q \quad \text{for } m=0,\ M-1, \qquad \text{(C-10)}$$

$$\nu_M = 1 - Mq, \qquad \text{(C-11)}$$

$$\nu_m = 0 \quad \text{for } m > M, \qquad \text{(C-12)}$$

where

$$M = \text{greatest integer } [1/q] \ . \qquad \text{(C-13)}$$

The proof of this goes as follows:

(i) If $\nu_o = q$, go to step (iv)

(ii) If $\nu_o < q$ and $\nu_o + \nu_1 = b \geqslant q$, put $\nu_o = q$ and $\nu_1 = b - q$. This will decrease Σ . Then go to step (iv).

(iii) If $\nu_o + \nu_1 = b < q$, put $\nu_o = b$, $\nu_1 = 0$ and reduce and other ν_m's to make $\nu_o = q$. This again will decrease Σ. Then go to step (iv).

(iv) Now work on ν_1 and ν_2 as we worked on ν_o and ν_1.

(v) Continue on with ν_2 and ν_3, etc.

We then have

$$\sigma^2/\ell^2 \geqslant \sum_{m=o}^{M-1} \frac{m^2 q}{4} + \frac{M^2}{4} [1 - Mq], \qquad \text{(C-14)}$$

which we can write in closed form as

$$\sigma^2/\ell^2 = r(q) = \frac{(M-1)M(2M-1)}{24} q + \frac{M^2}{4} [1 - Mq]. \qquad \text{(C-15)}$$

We now need only show that this function is monotonically decreasing for $0 < q \leqslant 1$, and we have implicitly defined $z(\sigma)$.

First let $q = \frac{1}{M}$. Then

$$r(q) = \frac{(M-1)(2M-1)}{24}$$

which increases as q decreases.

We can write Equation C-14 as

$$r(q) = \frac{M^2}{4} + qM\left[\frac{-4M^2 - 3M + 1}{24}\right] \qquad \text{(C-16)}$$

For $\frac{1}{M+1} < q \leqslant \frac{1}{M}$, M is fixed in Equation C-16 and r(q) varies only as a constant times q. Since

$$-4M^2 - 3M + 1 < 0 \quad \text{for } q \leqslant 1,$$

we have that r(q) increases as q decreases from $\frac{1}{M}$ to $\frac{1}{M+1}$.

We should point out that $r(\cdot)$ is not actually achievable by a $u(\cdot)$. In particular, there is a jump of q at zero and a jump of $q/2$ at $\ell/2$ so the interval $(-\ell/4, 3\ell/4)$ would have measure $3q/2$. Thus z is only a lower bound.

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